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Finding central nodes in large networks

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Google PageRank

- PageRank r_i of page $i = 1, \ldots, n$ is defined as:

$$r_i = \sum_{j\,:\,j \to i} \frac{\alpha}{d_j} r_j + (1-\alpha) q_i, \quad i = 1, \ldots, n$$

- $d_j = \#$ out-links of page j
- $\alpha \in (0,1)$, *damping factor* originally 0.85
- $q_i \geqslant 0$, $\sum_i q_i = 1$, originally, $q_i = 1/n$.

Easily bored surfer model

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- The page is important if many important pages link to it!

Linear equation and eigenvector problem

$$\mathbf{r} = \alpha \mathbf{r} P + (1-\alpha)\mathbf{q}$$

$$\mathbf{r} = \mathbf{r}\left[\alpha P + \frac{1-\alpha}{n}\mathbf{1}^t \mathbf{q}\right] \qquad \text{eigenvector problem}$$

$$\mathbf{r} = (1-\alpha)\mathbf{q}[I - \alpha P]^{-1} = (1-\alpha)\mathbf{q}\sum_{t=0}^{\infty} \alpha^t P^t.$$

Matrix expansion

$$\mathbf{r} = \alpha \mathbf{r} P + (1-\alpha)\mathbf{q}$$

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- Computation by matrix iterations:

$$\begin{aligned}\mathbf{r}^{(0)} &= (1/n, \ldots, 1/n)\\ \mathbf{r}^{(k)} &= \alpha \mathbf{r}^{(k-1)} P + (1-\alpha)\mathbf{q}\\ &= \mathbf{r}^{(0)}\alpha^k P^k + (1-\alpha)\mathbf{q}\sum_{t=0}^{k-1}\alpha^t P^t\end{aligned}$$

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- Exponentially fast convergence due to $\alpha \in (0,1)$
- Matrix iterations are used to compute PageRank in practice Langville&Meyer 2004, Berkhin 2005

Plan

- Part I: Centrality & computaitonal aspects
- Part II: PageRank

Effect of adding or removing links

$$r_i = (1-\alpha)q_i + (1-\alpha)\sum_{j=1}^{n} q_j \sum_{t=1}^{\infty} \alpha^t (P^t)_{ji}$$

The influence of the nodes on the PageRank of node i decreases *exponentially* with the distance from i.

(X_t) – Markov chain with transition matrix P.

$$\sum_{t=1}^{\infty} \alpha^t (P^t)_{ji} = \sum_{t=1}^{\infty} \alpha^t E_j \mathbf{1}[X_t = i] = E_j[\#\text{ visits to } i \text{ before a jump}]$$

$$= P_j(\text{reach } i \text{ before a jump}) E_i[1 + (\#\text{ returns to } i \text{ before a jump})]$$

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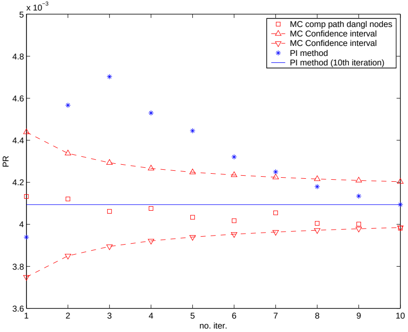
- Influence of out-degrees is very limited (Avrachenkov&L 2006)
- Your best PageRank boosting strategy?

Monte-Carlo computations

- Random walk from each node, length *Geometric*$(1-\alpha)$
- Compute the average number of visits to i



Avrachenkov, L, Nemirovsky, Osipova 2007

The influence of α

$$\mathbf{r} = (1-\alpha)\mathbf{q}\sum_{t=0}^{\infty}\alpha^t P^t$$

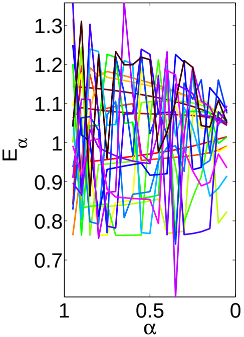


Figure: $-\log$(PageRank) for top-20 Dutch Wiki pages

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- $\alpha \approx 1$

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- Langville & Meyer 2004, Baeza-Yates, Boldi & Castillo 2006

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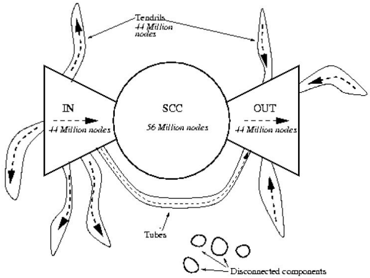

Figure: Broder et al. 2000

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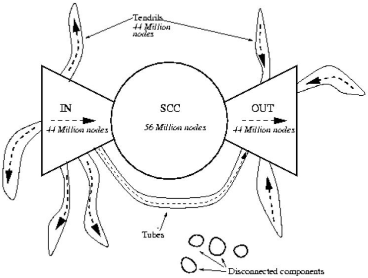

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- Choose $\alpha = 1/2$ to balance the components (Avrachenkov, L & Kim 2006)

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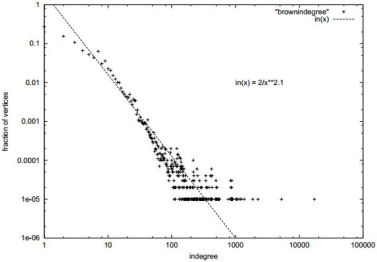
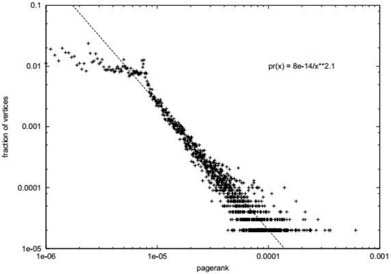
- Straight line on the log-log scale

Power law of PageRank

Pandurangan, Raghavan, Upfal 2002.





Stochastic model for PageRank

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- Rescale: $R_i = n r_i$, $Q_i \to n(1-\alpha)q_i$ so that $E(R) = 1$
- **Idea:** model R as a solution of stochastic equation (Volkovich&L 2010):

$$R \stackrel{d}{=} \alpha \sum_{j=1}^{N} \frac{1}{D_j} R_j + Q$$

- N: in-degree of the randomly chosen page
- D: out-degree of page that links to the randomly chosen page
- R_j is distributed as R; N, D, R_j are independent
- Denote $C_j = \alpha / D_j$.

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Results for stochastic recursion

$$R \stackrel{d}{=} \sum_{j=1}^{N} C_j R_j + Q$$

Theorem (Volkovich&L 2010)

If $P(Q > x) = o(P(N > x))$, *then the following are equivalent:*

- $P(N > x) \sim L(x)x^{-\gamma}$ *as* $x \to \infty$,
- $P(R > x) \sim aL(x)x^{-\gamma}$ *as* $x \to \infty$, *where* $a = (E[C])^{\gamma}(1 - \mathbb{E}[N]\mathbb{E}[C^{\gamma}])^{-1}$

- Here $a \sim b$ means $a/b \to 1$
- Note that $E[C] = 1/E(N)$, the role of out-degrees is minimal!

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- However, this does not completely explain the behavior of PageRank in networks because the recursion implicitly assumes an underlying *tree* structure.
- We now want to extend the result to random graphs!

Bi-directed degree sequence

- Directed graph on n nodes $V = \{v_1, \ldots, v_n\}$.
- Extended bi-degree sequence $(\mathbf{N}_n, \mathbf{D}_n, \mathbf{C}_n, \mathbf{Q}_n) = \{(N_i, D_i, C_i, Q_i) : 1 \leqslant i \leqslant n\}$

$$L_n = \sum_{i=1}^{n} N_i = \sum_{i=1}^{n} D_i$$

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- Assumption 1. Existence of certain limits in the spirit of the weak law of large numbers, including $\frac{1}{n}\sum_{i=1}^{n} D_i^2$ to be bounded in probability (finite variance of the out-degrees).
- Assumption 2. In a sequence of random graphs of growing size, the empirical probabilities $P(D_i = k)$ converge to certain distributions.
- Example: Chen&Olvera-Cravioto 2013

Directed Configuration Model (DCM)

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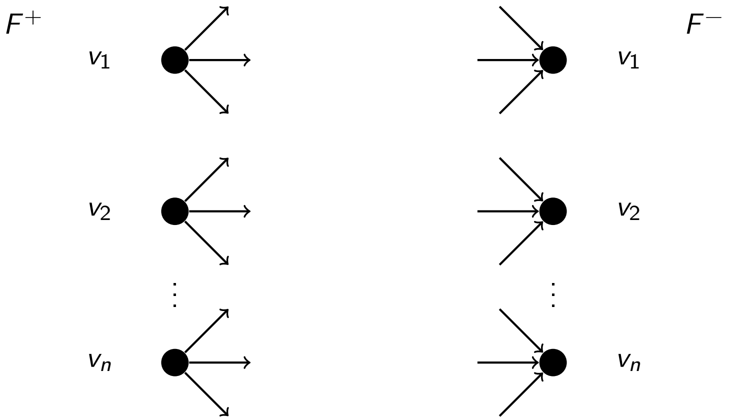

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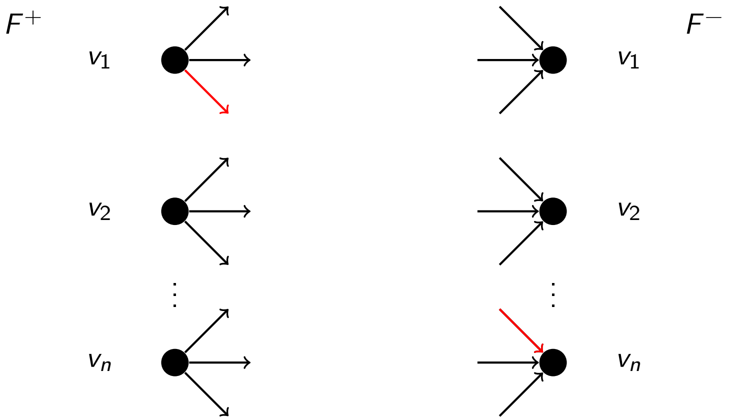

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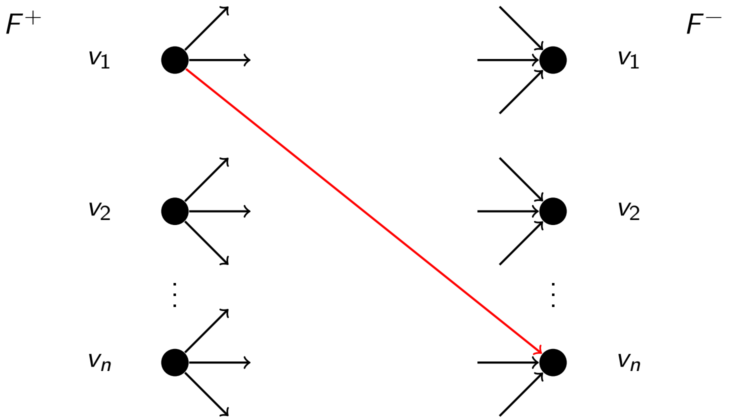

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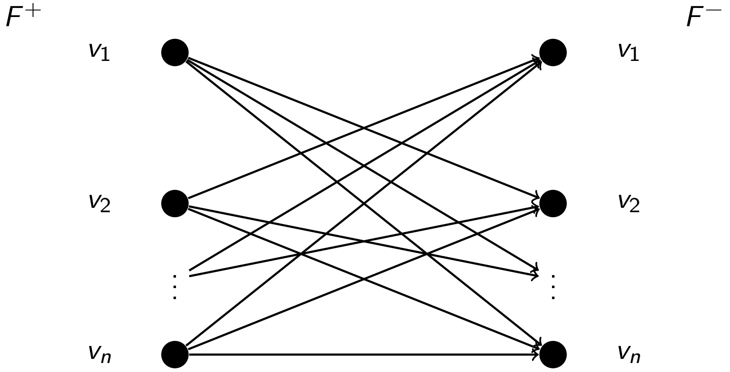

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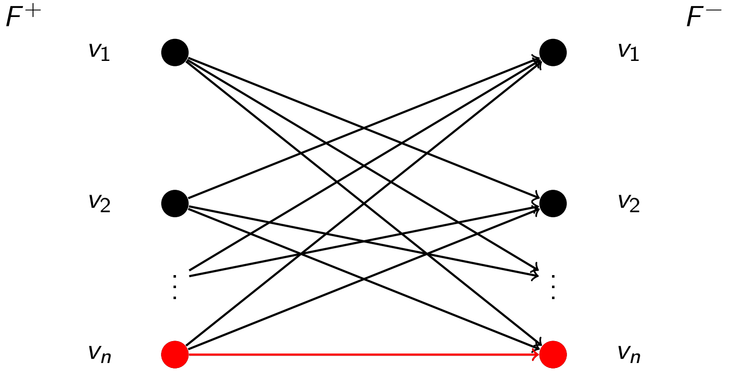

We keep self-loops and double edges.
The result is a multi-graph

PageRank in the DCM

Chen, L, Olvera-Cravioto 2014

- $M = M(n) \in \mathbb{R}^{n \times n}$ is related to the adjacency matrix of the graph:

$$M_{i,j} = \begin{cases} s_{ij} C_i, & \text{if there are } s_{ij} \text{ edges from } i \text{ to } j, \\ 0, & \text{otherwise.} \end{cases}$$

- $Q \in \mathbb{R}^n$ is a personalization vector
- We are interested in the distribution of one coordinate, $R_1^{(n)}$, of the vector $\mathbf{R}^{(n)} \in \mathbb{R}^n$ defined by

$$\mathbf{R}^{(n)} = \mathbf{R}^{(n)} M + Q$$

Original and size-biased distribution

- Given the extended bi-degree sequence $(\mathbf{N}_n, \mathbf{D}_n, \mathbf{C}_n, \mathbf{Q}_n)$:
- Empirical distribution for the root node's parameters:

$$F_n^*(m, q) := \frac{1}{n} \sum_{k=1}^{n} 1(N_k \leqslant m, Q_k \leqslant q),$$

converges to $F^*(m, q) := P(\mathcal{N}_0 \leqslant m, \mathcal{Q}_0 \leqslant q)$

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- Empirical distribution for a node that has a out-link to any arbitrary node (size-biased by out-degree)

$$F_n(m, q, x) := \sum_{k=1}^{n} 1(N_k \leqslant m, Q_k \leqslant q, C_k \leqslant x) \frac{D_k}{L_n}$$

converges to $F(m, q, x) := P(\mathcal{N} \leqslant m, \mathcal{Q} \leqslant q) P(\mathcal{C} \leqslant x)$.

Main result

Chen, L, Olvera-Cravioto 2016

$$\mathcal{R} \stackrel{\mathcal{D}}{=} \sum_{j=1}^{\mathcal{N}} \mathcal{C}_j \mathcal{R}_j + \mathcal{Q},$$

- Let $\mathcal{R}$ denote the *endogenous* solution to the SFPE above.
- The *endogenous* solution is the limit of iterations of the recursion starting, say, from $R_0 = \mathbf{1}$.
- Main result:

$$R_1^{(n)} \Rightarrow \mathcal{R}^*, \qquad n \to \infty,$$

where $\Rightarrow$ denotes weak convergence and $\mathcal{R}^*$ is given by

$$\mathcal{R}^* := \sum_{j=1}^{\mathcal{N}_0} \mathcal{C}_j \mathcal{R}_j + \mathcal{Q}_0,$$

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- 2. Coupling with a tree. Construct a coupling of the DCM graph and a “thorny branching tree” (TBT). The coupling between the graph and the TBT will hold for a number of generations in the tree that is logarithmic in n.

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- Three steps, three entirely different techniques.
- 1. Finite approximation. PageRank is accurately approximated by a finite number of matrix iterations.
- 2. Coupling with a tree. Construct a coupling of the DCM graph and a "thorny branching tree" (TBT). The coupling between the graph and the TBT will hold for a number of generations in the tree that is logarithmic in n.
- 3. Convergence to a weighted branching process. Show that the rank of the root node of the TBT converges weakly to the stated limit. Chen and Olvera-Cravioto (2014)

Matrix iterations

Under event $B_n = \left\{\max_{1\leqslant i\leqslant n}|C_i|D_i \leqslant \alpha,\ \frac{1}{n}\sum_{i=1}^{n}|Q_i| \leqslant H\right\}$

$$\left\|\mathbf{R}^{(n,k)} - \mathbf{R}^{(n,\infty)}\right\|_1 \leqslant \|\mathbf{r}_0\|_1\alpha^k + \sum_{i=0}^{\infty}\|\mathbf{Q}\|_1\alpha^{k+i} = |r_0|n\alpha^k + \|\mathbf{Q}\|_1\frac{\alpha^k}{1-\alpha}.$$

- We want to bound $|R_1^{(n,\infty)} - R_1^{(n,k)}|$
- The standard results on mixing times do not help to get rid of the factor n

Convergence for matrix iterations

- All nodes are symmetric!
- $E_n(|R_1^{(n,\infty)} - R_1^{(n,k)}|) = \frac{1}{n} E_n \left|\left|\mathbf{R}^{(n,k)} - \mathbf{R}^{(n,\infty)}\right|\right|_1$

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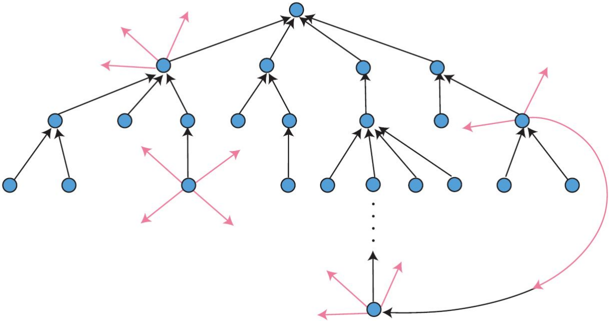
- $E_n(|R_1^{(n,\infty)} - R_1^{(n,k)}|) \leqslant |r_0|\alpha^k + \frac{\alpha^k}{n(1-\alpha)} \sum_i |Q_i|$
- Markov inequality:

$$P\left(\left| R_1^{(n,\infty)} - R_1^{(n,k)} \right| > x_n^{-1} \middle| B_n \right) = O\left(x_n \alpha^k\right)$$

- It is a weaker result than bounding $|R_1^{(n,\infty)} - R_1^{(n,k)}|$, but it is good enough
- Approximation of $R_1^{(n,\infty)}$ by $R_1^{(n,k)}$
- Next, approximate $R_1^{(n,k)}$ by the PageRank of a root of a tree with depth k

Coupling with branching tree

- We start with random node (node 1) and explore its neighbours, labeling the stubs that we have already seen
- τ – the number of generations of WBP completed before coupling breaks



Coupling with branching tree

Lemma (The Coupling Lemma)

Suppose $(\mathbf{N}_n, \mathbf{D}_n, \mathbf{C}_n, \mathbf{Q}_n)$ *satisfies WLLN,* $\mu = E(\mathcal{N}\mathcal{D})/E(\mathcal{D})$*. Then,*

- *for any* $1 \leqslant k \leqslant h \log n$ *with* $0 < h < 1/(2\log\mu)$*, if* $\mu > 1$*,*
- *for any* $1 \leqslant k \leqslant n^b$ *with* $b < 1/2$*, if* $\mu \leqslant 1$*,*

we have

$$P\left(\tau \leqslant k \middle| \Omega_n\right) = \begin{cases} O\left((n/\mu^{2k})^{-1/2}\right), & \mu > 1, \\ O\left((n/k^2)^{-1/2}\right), & \mu = 1, \\ O\left(n^{-1/2}\right), & \mu < 1, \end{cases}$$

as $n \to \infty$*.*

Remark: μ corresponds to the average number of offspring of a node in TBT.

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- Markov's inequality: $P(\tau = r) \leqslant \hat{Z}_r P_r = O\left(\mu^{2r} n^{-1}\right)$, $r \leqslant k$.

Main result

$$\mathcal{R} \stackrel{\mathcal{D}}{=} \sum_{j=1}^{\mathcal{N}} \mathcal{C}_j \mathcal{R}_j + \mathcal{Q},$$

- Let $\mathcal{R}$ denote the *endogenous* solution to the SFPE above.
- The *endogenous* solution is the limit of iterations of the recursion starting, say, from $R_0 = \mathbf{1}$.
- Main result:

$$R_1^{(n)} \Rightarrow \mathcal{R}^*, \qquad n \to \infty,$$

where $\Rightarrow$ denotes weak convergence and $\mathcal{R}^*$ is given by

$$\mathcal{R}^* := \sum_{j=1}^{\mathcal{N}_0} \mathcal{C}_j \mathcal{R}_j + \mathcal{Q}_0,$$

Numerical results-1

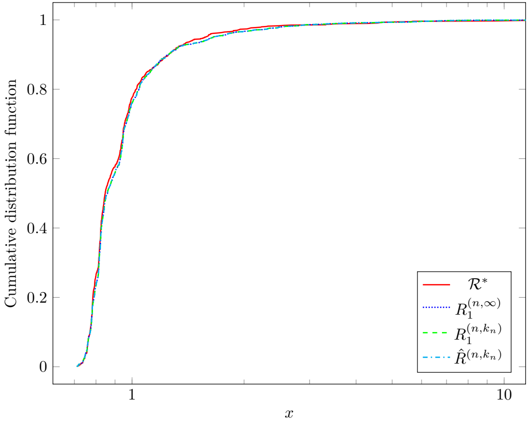


Figure: The empirical CDFs of 1000 samples of $\mathcal{R}^*$, $R_1^{(n,\infty)}$, $R_1^{(n,k_n)}$ and $\hat{R}^{(n,k_n)}$ for $n = 10000$ and $k_n = 9$.

Numerical results-2

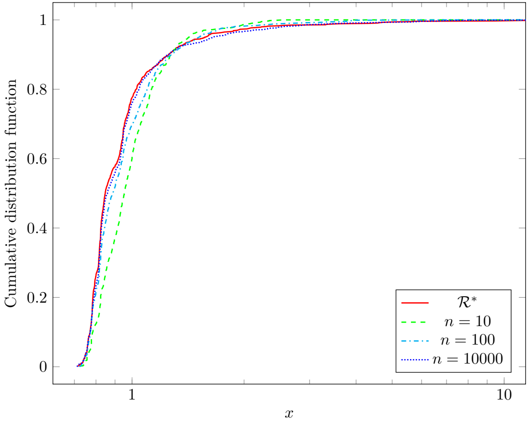
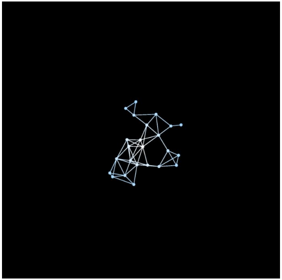


Figure: The empirical CDFs of 1000 samples of $\mathcal{R}^*$ and $R_1^{(n,\infty)}$ for $n = 10$, 100 and 10000.

Graph limits



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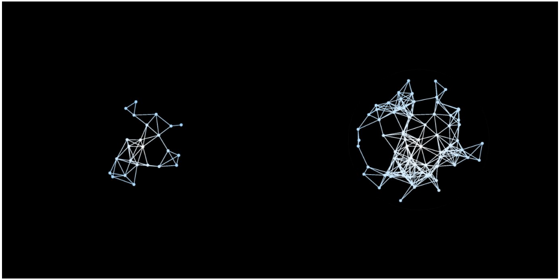

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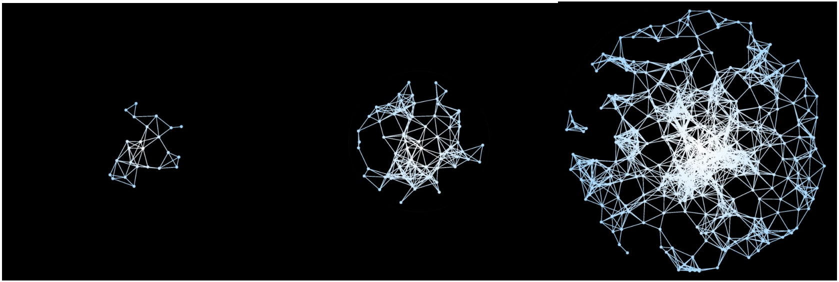
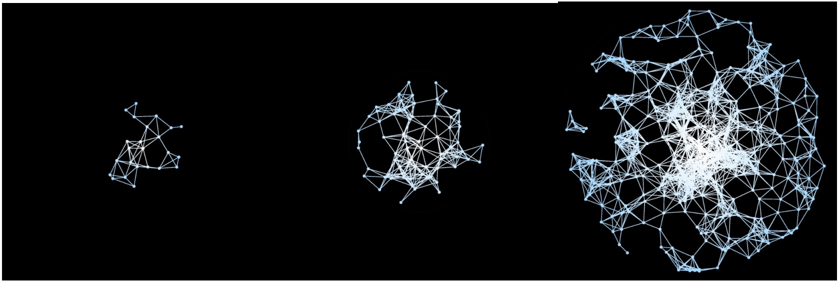


Graph limits



- Local weak convergence (Benjamini& Schramm 2001)

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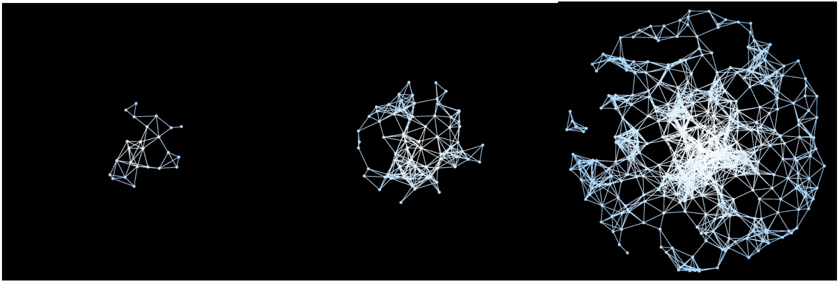

- Local weak convergence (Benjamini& Schramm 2001)
- Ongoing work vdHofstad, Garavaglia, L (2017)

Thank you!